ELTE RiskLab
Academia to industry

Quantum Time Series

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Fractional time series in Finance

Pricing and risk managing options

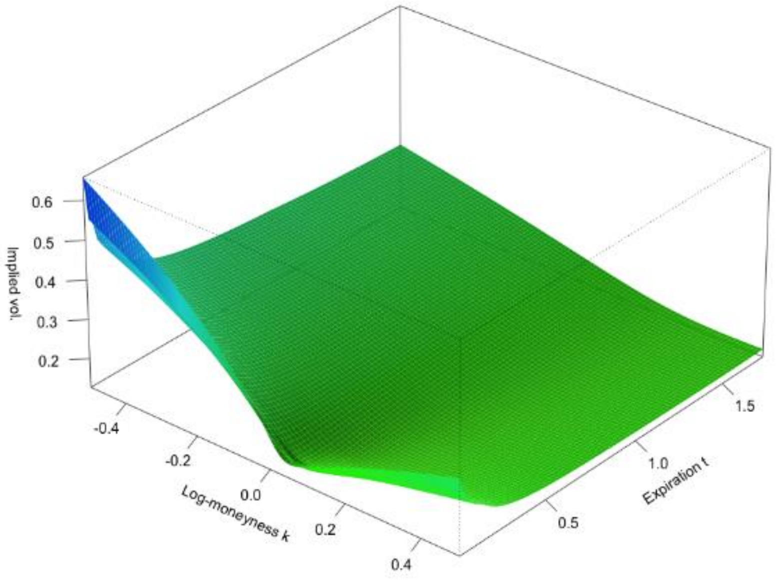


Figure 1.1: The S&P volatility surface as of June 20, 2013.

- Options modeled with stochastic volatility models
- Volatility is „rough” (Gatheral 2014)
- E.g.: Rough Fractional Stochastic Volatility (RFSV) model

$$dS_t = \mu_t S_t dt + \sigma_t S_t dZ_t$$

$$\sigma_t = \exp(X_t)$$

$$dX_t = \alpha(m - X_t)dt + \nu\, dB_t^H \qquad \text{fOU process}$$

- **Fractional Brownian motion** (fBm): B_t^H
  - H = Hurst exponent
  - Self similarity (mono-fractal property):

$$\{B_H(at), t \in \mathbb{R}\} \overset{law}{=} \{a^H B_H(t), t \in \mathbb{R}\}$$

Fractional Brownian motion

Definition: **fBm** is a centered Gaussian process with auto-covariance

$$cov(B_H(s), B_H(t)) = \frac{var(B_H(1))}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

Integral representations:

- Mandelbrot-Van Ness

$$B_H(t) = B_H(0) + \frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0}\left[(t-s)^{H-1/2} - (-s)^{H-1/2}\right] dB(s) + \int_0^t (t-s)^{H-1/2}\, dB(s)\right\}$$

- Molchan-Golosov

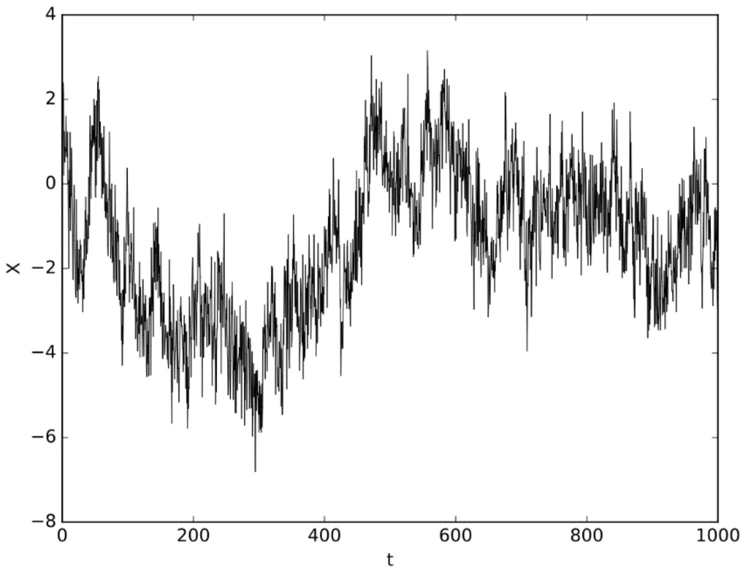
$$B_H(t) = \int_0^t K_H(t,s)\, dB(s) \qquad K_H(t,s) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}\, {}_2F_1\left(H-\frac{1}{2};\, \frac{1}{2}-H;\; H+\frac{1}{2};\, 1-\frac{t}{s}\right).$$

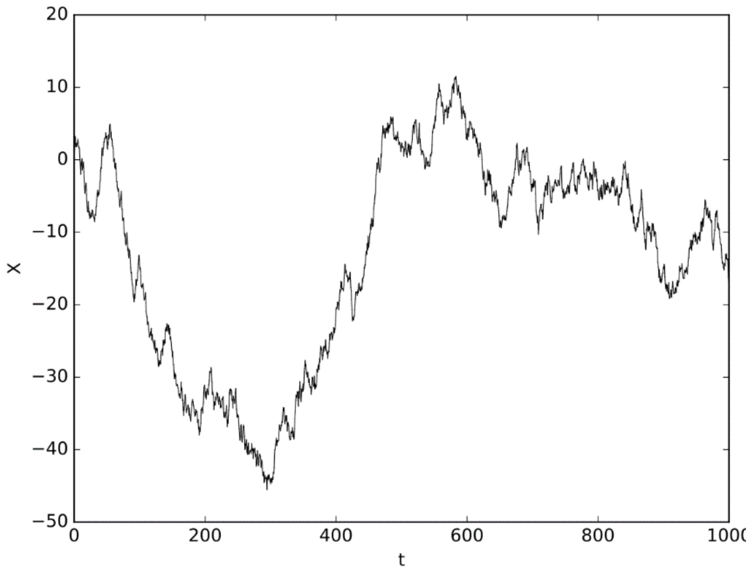
Trajectories

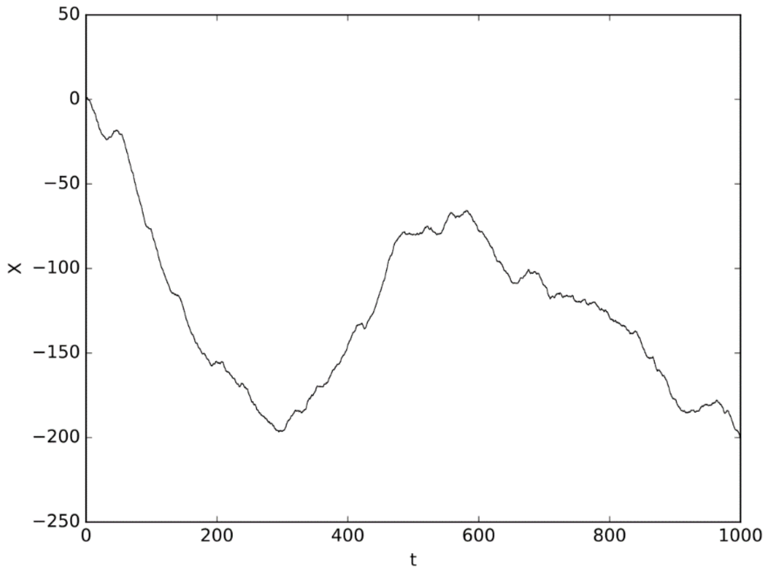
„rough" trajectories

standard Brownian motion trajectories

„smooth" trajectories







H =0.15

H=0.5

H=0.95

Simulation on discrete grid

Cholesky:

- Using the known covariance structure of fBm
- Scaling $O(n^3)$

Circulant method (Davies-Harte)

- Simulate the process increments first (Fractional Gaussian noise, fGn)
- fBm sample follows from cumulative sum
- Covariance of fGn has „circulant” (Toeplitz) structure
- Can be diagonalized with Fourier transform
- Scaling $O(n\ ln(n))$

$$\rho(k) = \mathbb{E}[\xi_1 \xi_{k+1}]$$
$$= \frac{1}{2n^{2H}} \left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right)$$

Covariance of the increments

Approximate methods

- Hybrid methods
- Kernel methods

Process of increments

Increments – Fractional Gaussian noise (fGn):

- Stationary
- Gaussian
- Autocorrelation (colored noise):

$$\rho(k) = \mathbb{E}[\xi_1 \xi_{k+1}] = \frac{1}{2n^{2H}} \left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H} \right)$$

- Asymptotically

$$\rho(k) \sim 2H(2H-1)\, k^{2H-2} \quad = \begin{cases} H < \frac{1}{2}: & \text{negative, fast decay, integrable} \\ H = \frac{1}{2}: & \text{zero, iid (White noise)} \\ H > \frac{1}{2}: & \text{positive, slow decay, non-integrable, long memory} \end{cases}$$

Algebraic decay

exponent: *2H-2*

Generalized Bernoulli and Fractional Binomial processes

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Generalized Bernoulli process with long-range dependence and fractional binomial distribution

Jeonghwa Lee

From the journal Dependence Modeling
https://doi.org/10.1515/demo-2021-0100

Definition: **Generalized Bernoulli Process**

We will define stationary process, $\{X_i, i \in \mathbb{N}\}$, where each X_i takes one of two possible outcomes, 0 or 1, with $P(X_i = 1) = p$, $P(X_i = 0) = 1 - p$, and

$$cov(X_i, X_j) = c'|i-j|^{2H-2}, i \neq j,$$

The integrated process is the **Fractional Binomial Process**

Define $B_n = \sum_{i=1}^{n} X_i$. It follows that $E(B_n) = np$, and as $n \to \infty$,

$$Var(B_n) \sim \begin{cases} \left(p(1-p) + \dfrac{c'}{2H-1}\right)n & H \in (0, 1/2), \\ c' n \ln n & H = 1/2, \\ \dfrac{c'}{2H-1}|n|^{2H}, & H \in (1/2, 1). \end{cases}$$

Spin chains

The **spin-1/2 XXZ spin chain** model is an example of a strongly correlated 1D quantum lattice system. The Hamiltonian is:

$$H = \sum_{j=1}^{N} \frac{1}{2}\left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right) + \Delta S_j^z S_{j+1}^z$$

Here S_j^+, S_j^- and S_j^z are the spin operators at site *j*, and Δ is the anisotropy parameter.

The phase diagram of the XXZ model exhibits different phases based on the value of Δ :

- **Isotropic AF ($\Delta = 1$):** Gapless. Critical behavior with algebraic decay of correlations
- **Gapless Phases ($-1 < \Delta < 1$):** Gapless. Critical behavior with different correlation exponents (Luttinger liquid)
- **Antiferromagnetic Phase ($\Delta < 1$):** Gapped. Long-range order with antiferromagnetic correlations (Néel order)
- **Ferromagnetic Phase ($\Delta > 1$):** Long-range order with ferromagnetic correlations

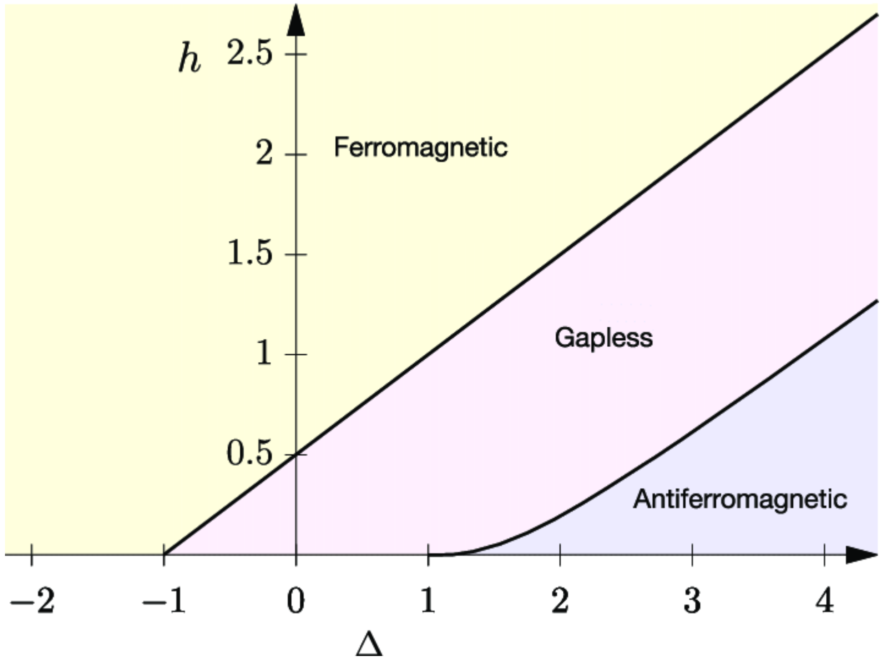
Gapless phase

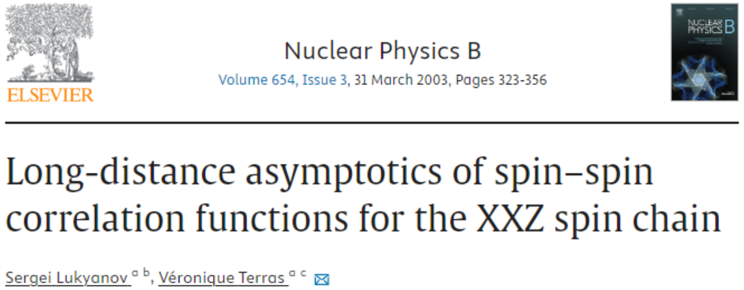
Asymptotics of correlation functions (Δ-dependent exponents):

$$C^x(n) = (-)^n \frac{A}{4n^\eta}\left(1 - \frac{B}{n^{4/\eta-4}}\right) - \frac{\tilde{A}}{4n^{\eta+1/\eta}}\left(1 + \frac{\tilde{B}}{n^{2/\eta-2}}\right) + \dots$$

$$C^z(n) = -\frac{1}{4\pi^2\eta\, n^2}\left(1 + \frac{\tilde{B}_z}{n^{4/\eta-4}}\frac{4-3\eta}{2-2\eta}\right) + (-)^n \frac{A_z}{4n^{1/\eta}}\left(1 - \frac{B_z}{n^{2/\eta-2}}\right) + \dots$$

$$\Delta = -\cos(\pi\eta), \qquad 0 < \eta < 1$$

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Nuclear Physics B
Volume 654, Issue 3, 31 March 2003, Pages 323-356

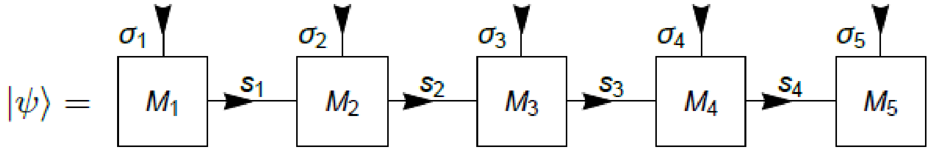
Long-distance asymptotics of spin–spin correlation functions for the XXZ spin chain

Sergei Lukyanov [a b], Véronique Terras [a c]

Matching

	Classical time series	Quantum chain
Integrated process	Fractional Binomial Process	Spin domain magnetization
Increment process	Generalized Bernoulli Process	Spin-1/2 quantum chain
Correlations	Power law (in time)	Power law (in space)
Fractal characteristics	Hurst exponent	Correlation exponents
Probability measure	Bernoulli probs conditional on history	Quantum ground state implied
Sample	Process trajectory	Spin chain configuration
Sampling algorithm	Cholesky, Circulant, Kernel	Quantum sampling from MPS

Matrix Product States



A **Matrix product state** (**MPS**) is a quantum state of many particles (in N sites), written in the following form:

$$|\Psi\rangle = \sum_{\{s\}} \mathrm{Tr}\left[A_1^{(s_1)} A_2^{(s_2)} \cdots A_N^{(s_N)}\right] |s_1 s_2 \ldots s_N\rangle,$$

where $A_i^{(s_i)}$ are complex, square matrices of order χ (this dimension is called local dimension). Indices s_i go over states in the computational basis. For qubits, it is $s_i \in \{0, 1\}$. For qudits (d-level systems), it is $s_i \in \{0, 1, \ldots, d-1\}$.

MPS Sampling

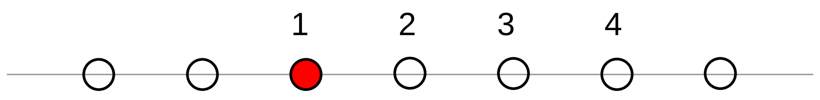
Perfect sampling with unitary tensor networks

Andrew J. Ferris and Guifre Vidal

Phys. Rev. B **85**, 165146 – Published 30 April 2012

Sampling sequentially from a known MPS state Ψ

1. Draw 1st spin:



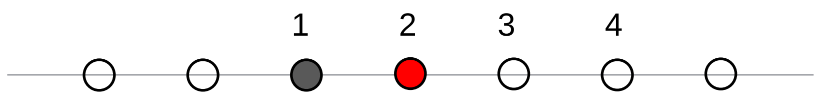
$$\rho_1 = \mathrm{Tr}_{2\ldots N}|\Psi\rangle\langle\Psi|$$

$$P(s_1) = \langle s_1|\rho_1|s_1\rangle$$

Draw s_1

Direction of sampling

2. Draw 2nd spin (conditioned on s_1):

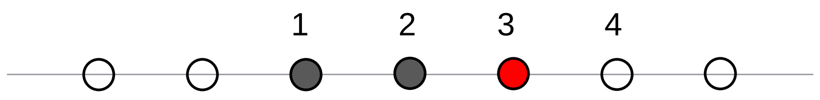


$$\rho_2(s_1) = \frac{1}{P(s_1)}\mathrm{Tr}_{3\ldots N}\langle s_1|\Psi\rangle\langle\Psi|s_1\rangle$$

$$P(s_2|s_1) = \langle s_2|\rho_2(s_1)|s_2\rangle$$

Draw s_2

3. Draw 3rd spin (conditioned on s_1, s_2):

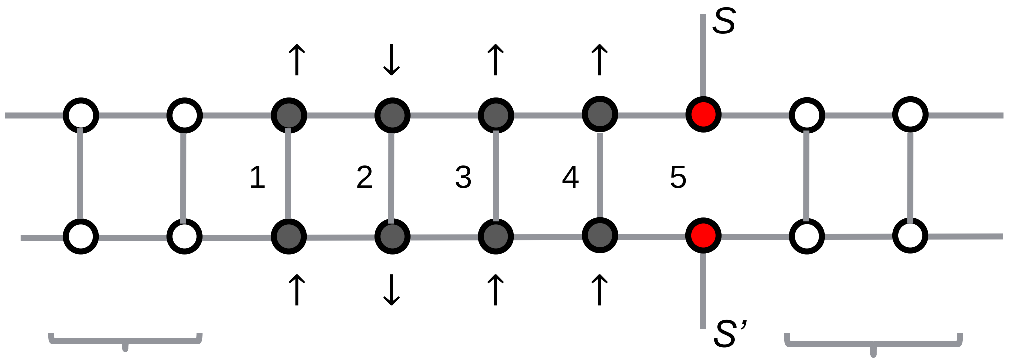


$$\rho_3(s_1, s_2) = \frac{1}{P(s_1, s_2)}\mathrm{Tr}_{4\ldots N}\langle s_1, s_2|\Psi\rangle\langle\Psi|s_1, s_2\rangle$$

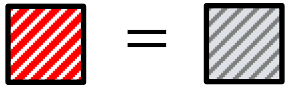
etc…

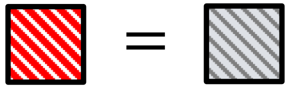
MPS sampling

This can be done effectively in the MPS formalism:



○ $= M = A^{\uparrow}|\uparrow\rangle + A^{\downarrow}|\downarrow\rangle$

 $= A^{\uparrow} \circ A^{\uparrow}$

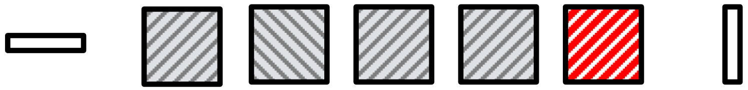
 $= A^{\downarrow} \circ A^{\downarrow}$

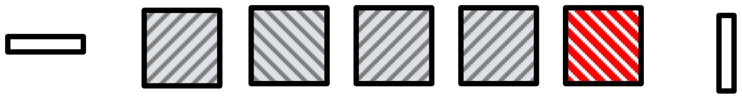
$(A^{\uparrow} \circ A^{\uparrow} + A^{\downarrow} \circ A^{\downarrow})$ ▯ $= \lambda_+$ ▯

○ $2 \cdot \chi \cdot \chi$

 $\chi^2 \cdot \chi^2$

▯ $\chi^2 \cdot 1$

$\rho_{\uparrow\uparrow} = \frac{1}{\lambda_+}$ 

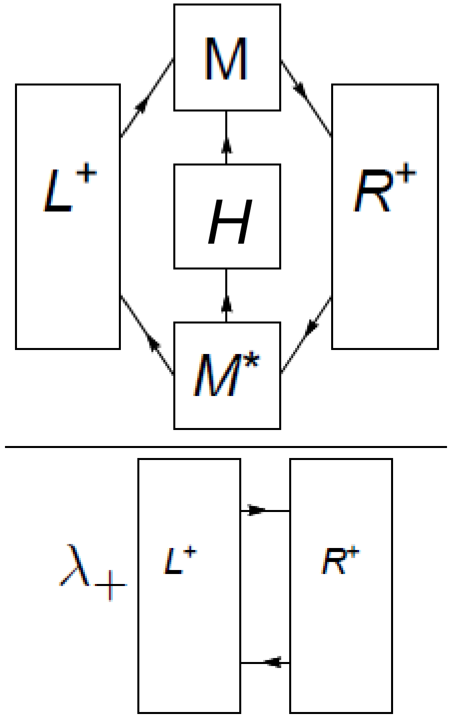
$\rho_{\downarrow\downarrow} = \frac{1}{\lambda_+}$ 

$P(\uparrow|history) = \frac{\rho_{\uparrow\uparrow}}{\rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow}}$ $\qquad$ $P(\downarrow|history) = \frac{\rho_{\downarrow\downarrow}}{\rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow}}$

Finding the MPS state

MPS is a variational Ansatz, it minimizes the energy for a suitable *M*.

Energy = $\langle H \rangle =$



H: Hamiltonian

M: MPS tensor

O = *MxM** transfer matrix

L: Left eigenvector of *O*

R: Right eigenvector of *O*

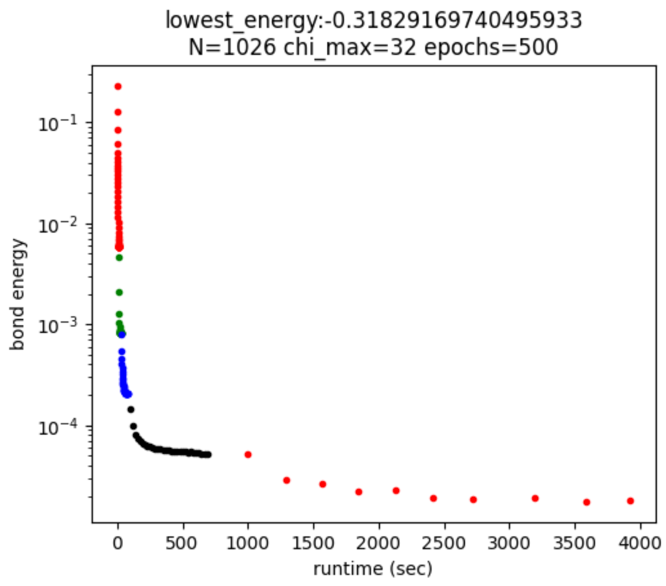
λ_+: Leading eigenvalue

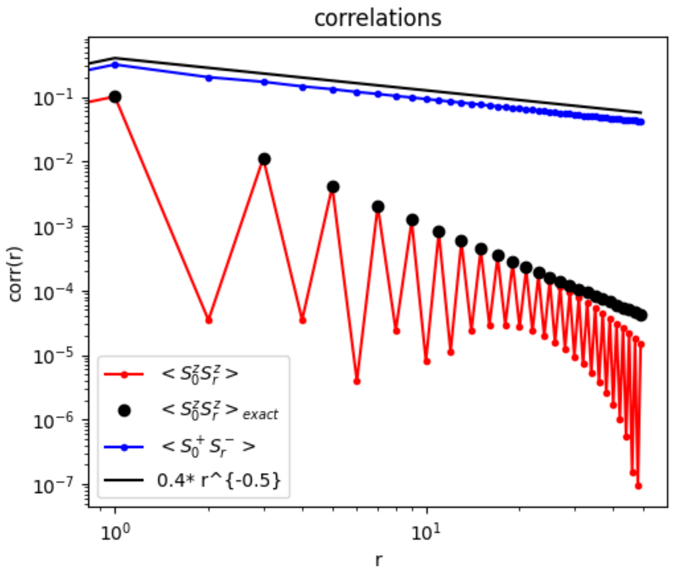
Finding the MPS state

- Density Matrix Renormalization Group (DMRG)

Alternatively:

- Convex optimization problem for the *M* tensor in a $2 \cdot \chi \cdot \chi$ dimensional space
- Gradient descent can be applied on a platform where Automatic Differentiation is implemented (e.g. TensorFlow)





H = 0.75

XXZ model $\Delta = 0$

H = 0

Thank you